

NON-VANISHING AND SIGN CHANGES OF HECKE EIGENVALUES FOR HALF-INTEGRAL WEIGHT CUSP FORMS

B. CHEN AND J. WU

In honor of J. G. van der Corput's 125th birthday

ABSTRACT. In this paper, we consider three problems about signs of the Fourier coefficients of a cusp form \mathfrak{f} with half-integral weight:

- The first negative coefficient of the sequence $\{\mathfrak{a}_{\mathfrak{f}}(tn^2)\}_{n \in \mathbb{N}}$,
- The number of coefficients $\mathfrak{a}_{\mathfrak{f}}(tn^2)$ of same signs,
- Non-vanishing of coefficients $\mathfrak{a}_{\mathfrak{f}}(tn^2)$ in short intervals and in arithmetic progressions,

where $\mathfrak{a}_{\mathfrak{f}}(n)$ is the n -th Fourier coefficient of \mathfrak{f} and t is a square-free integer such that $\mathfrak{a}_{\mathfrak{f}}(t) \neq 0$.

1. INTRODUCTION

Throughout we let $k \geq 1$ be an integer and assume $N \geq 4$ to be divisible by 4. Fix a real Dirichlet character χ modulo N . We write $\mathcal{S}_{k+1/2}(N, \chi)$ for the space of cusp forms of weight $k + 1/2$ for the group $\Gamma_0(N)$ with character χ . The space $\mathcal{S}_{3/2}(N, \chi)$ contains unary theta functions. Let $\mathcal{S}_{3/2}^*(N, \chi)$ be the orthogonal complement with respect to the Petersson scalar product of the subspace generated by these theta functions ([20, Section 4] and [3, Section 4]). For convenience, we put $\mathcal{S}_{k+1/2}^*(N, \chi) = \mathcal{S}_{k+1/2}(N, \chi)$ when $k \geq 2$. Each $\mathfrak{f} \in \mathcal{S}_{k+1/2}^*(N, \chi)$ has a Fourier expansion

$$(1.1) \quad \mathfrak{f}(z) = \sum_{n \geq 1} \mathfrak{a}_{\mathfrak{f}}(n) e^{2\pi i n z} \quad (\Im z > 0).$$

Let $\mathfrak{f} \in \mathcal{S}_{k+1/2}^*(N, \chi_0)$ be a cusp form with trivial character χ_0 , square-free level and real coefficients $\mathfrak{a}_{\mathfrak{f}}(n)$. Suppose that \mathfrak{f} lies in the plus space, that is, $\mathfrak{a}_{\mathfrak{f}}(n) = 0$ when $(-1)^k n \equiv 2, 3 \pmod{4}$, see [10, 8]. Bruinier and Kohnen [2] gave the conjectures

$$(1.2) \quad \lim_{x \rightarrow \infty} \frac{|\{n \leq x : \mathfrak{a}_{\mathfrak{f}}(n) \geq 0\}|}{|\{n \leq x : \mathfrak{a}_{\mathfrak{f}}(n) \neq 0\}|} = \frac{1}{2}$$

and

$$(1.3) \quad \lim_{x \rightarrow \infty} \frac{|\{|d| \leq x : d \text{ fundamental discriminant, } \mathfrak{a}_{\mathfrak{f}}(|d|) \geq 0\}|}{|\{|d| \leq x : d \text{ fundamental discriminant, } \mathfrak{a}_{\mathfrak{f}}(|d|) \neq 0\}|} = \frac{1}{2}$$

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with empirical evidence, which may be, however, out of present reach. Alternatively, they considered the sign changes of $\mathfrak{a}_{\mathfrak{f}}(n)$ when n runs over specific sets of integers, such as $\{tn^2\}_{n \in \mathbb{N}}$, $\{tp^{2\nu}\}_{\nu \in \mathbb{N}}$ and $\{tn_t^2\}_{t \text{ square-free}}$. Here t is a positive square-free integer such that $\mathfrak{a}_{\mathfrak{f}}(t) \neq 0$, p denotes any fixed prime and n_t is an integer determined by t (cf. [2, Theorems 2.1 and 2.2]). In particular, they proved that the sequence $\{\mathfrak{a}_{\mathfrak{f}}(tn^2)\}_{n \in \mathbb{N}}$ for a fixed square-free t has infinitely many sign changes. Recently, Kohnen, Lau and Wu [9] have proved some quantitative results on the number of sign changes in this sequence.

In this paper, we shall consider the following problems:

- The first negative coefficient of the sequence $\{\mathfrak{a}_{\mathfrak{f}}(tn^2)\}_{n \in \mathbb{N}}$;
- The number of coefficients $\mathfrak{a}_{\mathfrak{f}}(tn^2)$ of same signs;
- Non-vanishing of coefficients $\mathfrak{a}_{\mathfrak{f}}(tn^2)$ in short intervals and in arithmetic progressions.

Denote by $n_{\mathfrak{f}}$ the smallest integer n such that

$$(1.4) \quad \mathfrak{a}_{\mathfrak{f}}(tn^2) < 0 \quad \text{and} \quad (n, N/2) = 1.$$

The first aim of this paper is to give an upper bound of $n_{\mathfrak{f}}$. By using the Shimura lift of \mathfrak{f} [20] and developing the method of [12, 13], we can prove the following result.

Theorem 1. *Let $k \geq 1$ be an integer, $N \geq 4$ an integer divisible by 4 and χ be a real Dirichlet character modulo N . Suppose that $\mathfrak{f} \in \mathcal{S}_{k+1/2}^*(N, \chi)$ is a Hecke eigenform and $t \geq 1$ is a square-free integer such that $\mathfrak{a}_{\mathfrak{f}}(t) > 0$. Assume that its Shimura lift is not of CM type. In the above notation, we have*

$$n_{\mathfrak{f}} \ll (k^2 N)^{9/20},$$

where the implied constant is absolute.

Remark 1. The exponent $\frac{9}{20}$ can be reduced to $\frac{3}{8}$ if we do more numerical computation as in [16].

In order to study the number of coefficients $\mathfrak{a}_{\mathfrak{f}}(tn^2)$ of same signs, we introduce the counting functions:

$$(1.5) \quad \begin{cases} \mathcal{N}_{\mathfrak{f}}^*(x) := \sum_{\substack{n \leq x \\ \mathfrak{a}_{\mathfrak{f}}(tn^2) \neq 0}} 1, \\ \mathcal{N}_{\mathfrak{f}}^+(x) := \sum_{\substack{n \leq x \\ \mathfrak{a}_{\mathfrak{f}}(tn^2) > 0}} 1, \\ \mathcal{N}_{\mathfrak{f}}^-(x) := \sum_{\substack{n \leq x \\ \mathfrak{a}_{\mathfrak{f}}(tn^2) < 0}} 1. \end{cases}$$

By using the method of [17] based on multiplicative function theory, we establish the following result.

Theorem 2. *Let $k \geq 1$ be an integer, $N \geq 4$ an integer divisible by 4 and χ be a real Dirichlet character modulo N . Suppose that $\mathfrak{f} \in \mathcal{S}_{k+1/2}^*(N, \chi)$ is a Hecke eigenform and $t \geq 1$ is a square-free integer such that $\mathfrak{a}_{\mathfrak{f}}(t) \neq 0$. Assume that its Shimura lift is not of CM type.*

(i) *For any $\varepsilon > 0$, we have*

$$(1.6) \quad \mathcal{N}_{\mathfrak{f}}^*(x) = \rho_{\mathfrak{f}} x \left\{ 1 + O_{\mathfrak{f}, \varepsilon} \left((\log x)^{-1/4+\varepsilon} \right) \right\}$$

for $x \rightarrow \infty$, where $\delta_{\mathfrak{f}}(n)$ is the characteristic function of the integers n such that $\mathfrak{a}_{\mathfrak{f}}(tn^2) \neq 0$ and

$$(1.7) \quad \rho_{\mathfrak{f}} := \prod_p (1 - p^{-1}) \sum_{\nu \geq 0} \delta_{\mathfrak{f}}(p^{\nu}) p^{-\nu} > 0.$$

(ii) *For $x \rightarrow \infty$, we have*

$$(1.8) \quad \mathcal{N}_{\mathfrak{f}}^{\pm}(x) = \frac{1}{2} \rho_{\mathfrak{f}} x \left\{ 1 + O_{\mathfrak{f}, \varepsilon} \left((\log x)^{-1/4+\varepsilon} \right) \right\}.$$

Here the implied constants depend on \mathfrak{f} and ε .

Remark 2. (i) If $N/2$ is square-free, the assumption of a non-CM Shimura lift in Theorem 2 will automatically hold and hence can be omitted.

(ii) This theorem shows that the variant of (1.2) for $\{\mathfrak{a}_{\mathfrak{f}}(tn^2)\}_{n \in \mathbb{N}}$ holds and improves Theorem 2 of Kohnen, Lau and Wu [9].

In order to measure the non-vanishing of $\mathfrak{a}_{\mathfrak{f}}(tn^2)$, we introduce, as in [19],

$$i_{\mathfrak{f}}(n) := \max\{j \geq 1 : \mathfrak{a}_{\mathfrak{f}}(t(n+i)^2) = 0 \text{ for } 0 < i \leq j\}$$

with the convention that $\max \emptyset = 0$. We hope to get a non-trivial bound of type

$$i_{\mathfrak{f}}(n) \ll_{\mathfrak{f}, \theta} n^{\theta}$$

for some $\theta < 1$ and all $n \geq 1$. Clearly a stronger form of the problem is to find y as small as possible (as a function of x , $y = x^{\theta}$ with $\theta < 1$) such that

$$\#\{x < n \leq x + y : \mu(n)^2 = 1 \text{ and } \mathfrak{a}_{\mathfrak{f}}(tn^2) \neq 0\} \gg_{\mathfrak{f}, \theta} y,$$

where $\mu(n)$ is the Möbius function and the implied constant can depend on \mathfrak{f} and θ .

We have the following result.

Theorem 3. *Let $k \geq 1$ be an integer, $N \geq 4$ an integer divisible by 4 and χ be a real Dirichlet character modulo N . Suppose that $\mathfrak{f} \in \mathcal{S}_{k+1/2}^*(N, \chi)$ is a Hecke eigenform and $t \geq 1$ is a square-free integer such that $\mathfrak{a}_{\mathfrak{f}}(t) \neq 0$. Assume that its Shimura lift is not of CM type.*

(i) *For every $\varepsilon > 0$, $x \geq x_0(\mathfrak{f}, \varepsilon)$ and $y \geq x^{7/17+\varepsilon}$, we have*

$$\left| \{x < n \leq x + y : \mu(n)^2 = 1 \text{ and } \mathfrak{a}_{\mathfrak{f}}(tn^2) \neq 0\} \right| \gg_{\mathfrak{f}, \varepsilon} y.$$

In particular for any $\varepsilon > 0$ and all $n \geq 1$, we have

$$i_{\mathfrak{f}}(n) \ll_{\mathfrak{f}, \varepsilon} n^{7/17+\varepsilon}.$$

(ii) *For every $\varepsilon > 0$, $x \geq x_0(\mathfrak{f}, \varepsilon)$, $y \geq x^{17/38+100\varepsilon}$ and $1 \leq a \leq q \leq x^{\varepsilon}$ with $(a, q) = 1$, we have*

$$\left| \{x < n \leq x + y : \mu(n)^2 = 1, n \equiv a \pmod{q} \text{ and } \mathfrak{a}_{\mathfrak{f}}(tn^2) \neq 0\} \right| \gg_{\mathfrak{f}, \varepsilon} y/q.$$

Here the implied constants depend on \mathfrak{f} and ε .

Remark 3. This theorem can be proved with the help of the \mathcal{B} -free number method as in [11, 22]. Our principal tools are some estimates for multiple exponential sums (see [11, Proposition 5] and [22, Proposition 2.1]). J. G. van der Corput is a pioneer on this domain. On the occasion of his 125th birthday, it is a pleasure for us to write this paper to commemorate the outstanding contribution he made to the number theory.

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2. BACKGROUND

For $\mathfrak{f} \in \mathcal{S}_{k+1/2}^*(N, \chi)$, let $t \geq 1$ be a square-free integer such that $\mathfrak{a}_{\mathfrak{f}}(t) \neq 0$. The Shimura correspondence [20] lifts \mathfrak{f} to a cusp form f_t of weight $2k$ for the group $\Gamma_0(N/2)$ with character χ^2 . Also it gives a vital relation between their Fourier coefficients,

$$(2.1) \quad a_{f_t}(n) := \sum_{d|n} \chi_{t,N}(d) d^{k-1} \mathfrak{a}_{\mathfrak{f}}\left(t \frac{n^2}{d^2}\right),$$

where $\chi_{t,N}$ denotes the character

$$(2.2) \quad \chi_{t,N}(d) := \chi(d) \left(\frac{(-1)^k t}{d} \right)$$

($\left(\frac{(-1)^k t}{d} \right)$ is Legendre's symbole) and

$$(2.3) \quad f_t(z) := \sum_{n \geq 1} a_{f_t}(n) e^{2\pi i n z} \quad (\Im z > 0).$$

(f_t is called the Shimura lift of \mathfrak{f} associated to t .) Furthermore, if \mathfrak{f} is a Hecke eigenform, then so is the Shimura lift of \mathfrak{f} . In fact, in this case,

$$(2.4) \quad f(z) := \mathfrak{a}_{\mathfrak{f}}(t)^{-1} f_t(z)$$

is a normalized Hecke eigenform independent of t (i.e. the first coefficient of Fourier is equal to 1), and the arithmetic function $n \mapsto \mathfrak{a}_{\mathfrak{f}}(tn^2)$ is multiplicative in the following sense (cf. [20, (1.18)]):

$$(2.5) \quad \mathfrak{a}_{\mathfrak{f}}(tm^2) \mathfrak{a}_{\mathfrak{f}}(tn^2) = \mathfrak{a}_{\mathfrak{f}}(t) \mathfrak{a}_{\mathfrak{f}}(tm^2 n^2) \quad \text{if } (m, n) = 1.$$

Write

$$(2.6) \quad \lambda_f(n) := \mathfrak{a}_{\mathfrak{f}}(t)^{-1} a_{f_t}(n) n^{-(2k-1)/2}, \quad \mathfrak{a}_{\mathfrak{f}}^*(n) := \mathfrak{a}_{\mathfrak{f}}(t)^{-1} \mathfrak{a}_{\mathfrak{f}}(tn^2) n^{-(k-1/2)}.$$

Clearly $\lambda_f(n)$ is the n -th normalized Fourier coefficient of f (i.e., $\lambda_f(1) = 1$) and the function $n \mapsto \mathfrak{a}_{\mathfrak{f}}^*(n)$ is multiplicative.

Further we introduce the following notation:

$$(2.7) \quad \mathcal{L} := \log(C_0 k N) \quad \text{and} \quad N_k := \prod_{p \leq \mathcal{L}^2 \text{ or } p | (N/2)} p,$$

where C_0 is an absolute large constant. Write

$$\left\{ \begin{array}{l} L(s, f) := \sum_{n \geq 1} \lambda_f(n) n^{-s}, \\ L(s, \mathfrak{a}_f^*) := \sum_{n \geq 1} \mathfrak{a}_f^*(n) n^{-s}, \\ L(s, \chi_{t,N}) := \sum_{n \geq 1} \chi_{t,N}(n) n^{-s}, \end{array} \right\} \quad \left\{ \begin{array}{l} L^b(s, f) := \sum_{n \geq 1}^b \lambda_f(n) n^{-s}, \\ L^b(s, \mathfrak{a}_f^*) := \sum_{n \geq 1}^b \mathfrak{a}_f^*(n) n^{-s}, \\ L^b(s, \chi_{t,N}) := \sum_{n \geq 1}^b \chi_{t,N}(n) n^{-s}, \end{array} \right.$$

where \sum^b means that the sum runs over square-free integers n satisfying $(n, N_k) = 1$.

Lemma 2.1. *Let $k \geq 1$ be an integer, $N \geq 4$ an integer divisible by 4 and χ be a real Dirichlet character modulo N . Suppose that $\mathfrak{f} \in \mathcal{S}_{k+1/2}^*(N, \chi)$ is a Hecke eigenform and $t \geq 1$ is a square-free integer such that $\mathfrak{a}_f(t) \neq 0$.*

(i) *We have*

$$(2.8) \quad L(s, f) = L(s + \tfrac{1}{2}, \chi_{t,N}) L(s, \mathfrak{a}_f^*),$$

for all $s \in \mathbb{C}$; and

$$(2.9) \quad L^b(s, f) = L^b(s + \tfrac{1}{2}, \chi_{t,N}) L^b(s, \mathfrak{a}_f^*) \mathcal{L}_f(s),$$

for $\sigma > \tfrac{1}{2}$, where the Dirichlet series $\mathcal{L}_f(s)$ converges absolutely for $\sigma > \tfrac{1}{2}$ and

$$(2.10) \quad \mathcal{L}_f(s) \asymp_\varepsilon 1 \quad (\sigma \geq \tfrac{1}{2} + \varepsilon).$$

Here the implied constant depends only on ε .

(ii) *For any $\varepsilon > 0$, we have*

$$(2.11) \quad L(\sigma + i\tau, \chi_{t,N})^{-1} \ll \varepsilon^{-1}$$

and

$$(2.12) \quad L^b(\sigma + i\tau, \chi_{t,N})^{-1} \ll \varepsilon^{-2}$$

for $\sigma \geq 1 + \varepsilon$ and $\tau \in \mathbb{R}$, where the implied constants are absolute.

(iii) *For any $\varepsilon > 0$, we have*

$$(2.13) \quad |L(\sigma + i\tau, \mathfrak{a}_f^*)| \ll_\varepsilon (N(k + |\tau|)^2)^{\max\{(1-\sigma)/2, 0\} + \varepsilon}$$

and

$$(2.14) \quad |L^b(\sigma + i\tau, \mathfrak{a}_f^*)| \ll_\varepsilon (N(k + |\tau|)^2)^{\max\{(1-\sigma)/2, 0\} + \varepsilon}$$

for $\sigma \geq \tfrac{1}{2} + \varepsilon$ and $\tau \in \mathbb{R}$, where the implied constants depend only on ε .

Proof. In view of the definition (2.6), the formula (2.1) is equivalent to

$$(2.15) \quad \lambda_f(n) = \sum_{d|n} \frac{\chi_{t,N}(d)}{\sqrt{d}} \mathfrak{a}_f^*\left(\frac{n}{d}\right).$$

Clearly this formula implies (2.8) for $\sigma > 1$. By analytic continuation, this relation is true for all $s \in \mathbb{C}$ since $L(s, f)$ and $L(s + \frac{1}{2}, \chi_{t,N})$ are entire.

Put $g(n) := \chi_{t,N}(n)\mu(n)^2/\sqrt{n}$ and $h(n) := \mathfrak{a}_f^*(n)\mu(n)^2$, where $\mu(n)$ is the Möbius function. Define the multiplicative function $\ell(n)$ by the relation

$$(2.16) \quad \lambda_f(n)\mu(n)^2 = (g * h * \ell)(n).$$

In view of (2.15), we have $\ell(p) = 0$ for all primes p . Next we shall prove that there is an absolute constant C such that

$$(2.17) \quad |\ell(p^\nu)| \leq C^\nu$$

for all primes p and all integers $\nu \geq 2$. In fact, the definition of $\ell(n)$ allows us to write

$$\begin{aligned} 0 &= \lambda_f(p^\nu)\mu(p^\nu)^2 = \sum_{\nu_1+\nu_2+\nu_3=\nu} g(p^{\nu_1})h(p^{\nu_2})\ell(p^{\nu_3}) \\ &= \ell(p^\nu) + \ell(p^{\nu-1})(g(p) + h(p)) + \ell(p^{\nu-2})g(p)h(p) \end{aligned}$$

for all primes p and all integers $\nu \geq 2$. This implies

$$(2.18) \quad |\ell(p^\nu)| \leq (|g(p)| + |h(p)| + |g(p)h(p)|) \max\{|\ell(p^{\nu-1})|, |\ell(p^{\nu-2})|\}.$$

In view of the definition of $g(n)$ and of $h(n)$ and [14, Lemma 6], we easily see that there is an absolute constant C such that

$$(2.19) \quad |g(p)| + |h(p)| + |g(p)h(p)| \leq C.$$

Clearly (2.17) follows immediately from (2.18) and (2.19) by a simple recurrence. Now by using (2.16), (2.17) and $\ell(p) = 0$, the formula (2.9) holds for $\sigma > \frac{1}{2}$ with

$$\mathcal{L}_f(s) := \sum_{\substack{n \geq 1 \\ (n, N_k)=1}} \ell(n)n^{-s} = \prod_{p \nmid N_k} \left(1 + \sum_{\nu \geq 2} \ell(p^\nu)p^{-\nu s}\right)$$

and we have $\mathcal{L}_f(s) \asymp_\varepsilon 1$ for $\sigma \geq \frac{1}{2} + \varepsilon$ thanks to the fact that $p \nmid N_k$ implies $p \geq \mathcal{L}^2 \geq (\log C_0)^2 \geq 100C^2$, since we have supposed that C_0 is a large constant. Here the implied constants in the \asymp_ε -symbol depend only on ε .

Next we prove the assertion (ii). Since $\chi_{t,N}^2 = \chi_0$, Theorem II.8.7 of [21] with $\chi = \chi_{t,N}$ gives us

$$L(\sigma, \chi_0)^3 |L(\sigma + i\tau, \chi_{t,N})|^4 |L(\sigma + i2\tau, \chi_0)| \geq 1$$

for $\sigma > 1$ and $\tau \in \mathbb{R}$. On the other hand, for $\sigma \geq 1 + \varepsilon$ and $\tau \in \mathbb{R}$ we have trivially

$$|L(\sigma, \chi_0)| + |L(\sigma + i2\tau, \chi_0)| \leq 2\zeta(1 + \varepsilon) \ll \varepsilon^{-1},$$

where the implied constant is absolute. The inequality (2.11) follows immediately.

For $\sigma > 1$, we have

$$L^\flat(s, \chi_{t,N}) = \prod_{p \nmid N_k} \left(1 + \frac{\chi_{t,N}(p)}{p^s}\right) = L(s, \chi_{t,N}) G_{\chi_{t,N}}(s),$$

where the Dirichlet series of

$$G_{\chi_{t,N}}(s) := \prod_{p|N_k} \left(1 - \frac{\chi_{t,N}(p)}{p^s}\right) \prod_{p \nmid N_k} \left(1 - \frac{\chi_{t,N}(p)^2}{p^{2s}}\right)$$

converges absolutely and so $G_{\chi_{t,N}}(s) \gg_{\varepsilon} N^{-\varepsilon}$ in the half-plane $\sigma \geq \frac{1}{2} + \varepsilon$ (with the implied constant depending only on ε) and $G_{\chi_{t,N}}(s) \gg \varepsilon$ for $\sigma \geq 1 + \varepsilon$ (here the implied constant is absolute). Now the inequality (2.12) follows immediately from (2.11).

Finally we treat the assertion (iii). Under our hypothesis, f is a newform of weight $2k$ for the group $\Gamma_0(N/2)$ with character χ^2 . Thus we have the convexity bound

$$(2.20) \quad L(s, f) \ll_{\varepsilon} (N(k + |\tau|)^2)^{\max\{(1-\sigma)/2, 0\} + \varepsilon}$$

for $\sigma \geq \frac{1}{2} + \varepsilon$ and $\tau \in \mathbb{R}$ (see [18, page 202, (1.22)] or [7, page 4, (1.12)]), where the implied constant depends only on ε .

For $\sigma > 1$, we have

$$L^{\flat}(s, f) = \prod_{p \nmid N_k} \left(1 + \frac{\lambda_f(p)}{p^s} \right) = L(s, f) G_f(s),$$

where the Dirichlet series of

$$G_f(s) := \prod_{p \mid N_k} \left(1 - \frac{\lambda_f(p)}{p^s} \right) \prod_{p \nmid N_k} \left(1 - \frac{\lambda_f(p)^2}{p^{2s}} \right)$$

converges absolutely and so $G_f(s) \ll_{\varepsilon} (kN)^{\varepsilon}$ in the half-plane $\sigma \geq \frac{1}{2} + \varepsilon$ (as $|\lambda_f(p)| \leq 2$ by Deligne's inequality [4]). Using the convexity bound (2.20), we can derive

$$(2.21) \quad L^{\flat}(s, f) \ll_{\varepsilon} (N(k + |\tau|)^2)^{\max\{(1-\sigma)/2, 0\} + \varepsilon}$$

for $\sigma \geq \frac{1}{2} + \varepsilon$ and $\tau \in \mathbb{R}$. Here the implied constants in the \ll_{ε} -symbol depend only on ε .

By (2.8), (2.11) and (2.20), we get (2.13). Similarly we can derive (2.14) from (2.9), (2.10), (2.12) and (2.21). \square

The second lemma will be needed in the proof of Theorem 2.

Lemma 2.2. *Let $k \geq 1$ be an integer, $N \geq 4$ an integer divisible by 4 and χ be a real Dirichlet character modulo N . Suppose that $\mathfrak{f} \in \mathcal{S}_{k+1/2}^*(N, \chi)$ is a Hecke eigenform and $t \geq 1$ is a square-free integer such that $\mathfrak{a}_{\mathfrak{f}}(t) > 0$. Assume that its Shimura lift f_t is not of CM type. Then for any $\varepsilon > 0$, there is a constant $x_0(\mathfrak{f}, \varepsilon)$ such that*

$$\sum_{\substack{p \leq x \\ \mathfrak{a}_{\mathfrak{f}}(tp^2) < 0}} 1 \geq \left(\frac{1}{2} - \varepsilon \right) \frac{x}{\log x}$$

for $x \geq x_0(\mathfrak{f}, \varepsilon)$.

Proof. Taking $n = p$ in (2.15), it follows that

$$\mathfrak{a}_{\mathfrak{f}}^*(p) = \lambda_f(p) - \frac{\chi_{t,N}(p)}{\sqrt{p}}.$$

In view of the hypothesis that $\mathfrak{a}_{\mathfrak{f}}(t) > 0$ and (2.6), we have

$$p > \varepsilon^{-2} \text{ and } \lambda_f(p) \leq -\varepsilon \Rightarrow \mathfrak{a}_{\mathfrak{f}}^*(p) < 0 \Rightarrow \mathfrak{a}_{\mathfrak{f}}(tp^2) < 0.$$

Thus

$$\sum_{\substack{p \leq x \\ \mathfrak{a}_f(tp^2) < 0}} 1 \geq \sum_{\substack{\varepsilon^{-2} < p \leq x \\ \lambda_f(p) \leq -\varepsilon}} 1.$$

Now the required inequality is an immediate consequence of the Sato-Tate conjecture (proved by Barnet-Lamb, Geraghty, Harris and Taylor [1]). \square

The next lemma comes from the first part of Theorem 15 in Serre [19], which is the key tool for the proof of Theorem 3.

Lemma 2.3. *Let g be any normalized Hecke eigenform of integral weight ≥ 2 and of level M . Suppose that $\ell(X) \in \mathbb{C}[X]$ is any polynomial. Write $a_g(n)$ for the n -th Fourier coefficient of g . If g is not of CM type, then*

$$(2.22) \quad \sum_{\substack{p \leq x \\ p \nmid M, a_g(p) = \ell(p)}} 1 \ll_{g, \ell, \varepsilon} \frac{x}{(\log x)^{5/4 - \varepsilon}}$$

holds for any $\varepsilon > 0$ and all $x \geq 2$, where the implied constant depends on g, ℓ and ε .

3. THE PROOF OF THEOREM 1

Let N_k be defined as in (2.7). Consider the summatory function

$$(3.1) \quad S_{\mathfrak{f}}(x) := \sum_{\substack{n \leq x \\ (n, N_k) = 1}} \mathfrak{a}_{\mathfrak{f}}^*(n) \mu(n)^2 \log \left(\frac{x}{n} \right) = \sum_{n \leq x}^b \mathfrak{a}_{\mathfrak{f}}^*(n) \log \left(\frac{x}{n} \right).$$

3.1. Upper bound for $S_{\mathfrak{f}}(x)$.

Proposition 1. *Under the condition of Theorem 1, we have*

$$(3.2) \quad S_{\mathfrak{f}}(x) \ll_{\varepsilon} (k^2 N)^{1/4 + \varepsilon} x^{1/2}$$

for all $x \geq 2$, where the implied constant depends on ε only.

Proof. The Perron formula (cf. [21, Theorem II.2.3]) gives

$$S_{\mathfrak{f}}(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} L^b(s, \mathfrak{a}_{\mathfrak{f}}^*) \frac{x^s}{s^2} ds.$$

Moving the line of integration $\Re s = 2$ to $\Re s = \frac{1}{2} + \varepsilon$ and applying the convexity bound (2.14) for $L^b(s, \mathfrak{a}_{\mathfrak{f}}^*)$, we obtain the required upper bound (3.2). \square

3.2. Two preliminary lemmas.

In order to establish the required lower bound for $S_{\mathfrak{f}}(x)$, we need two mean value theorems of multiplicative functions over friable (i.e. smooth) integers coprime with q . For $x \geq 1$, $y \geq 2$ and $q \in \mathbb{N}$, define

$$\Xi_q(x, y) := \sum_{\substack{n \leq x, (n, q) = 1 \\ P(n) \leq y}} \mu(n)^2 \quad \text{and} \quad \Xi_q(x) := \Xi_q(x, x),$$

where $P(n)$ is the largest prime factor of n .

The first lemma is a particular case of [13, Lemma 4.2] with $\kappa = 1$.

Lemma 3.1. *Let $U > 1$ be a fixed constant. For some suitable constant $C = C(U)$ depending only on U , we have*

$$(3.3) \quad \Xi_q(y^u, y) = \Pi_q y^u \rho(u) \left\{ 1 + O_U \left(\frac{L_q^{e+2}}{\sqrt{\log y}} \right) \right\}$$

uniformly for

$$(3.4) \quad q \geq 1, \quad y \geq \exp(2CL_q^{e+2}), \quad U^{-1} \leq u \leq U,$$

where $L_q := \log(\omega(q) + 3)$, $\omega(q)$ is the number of distinct prime factors of q , $\varphi(q)$ is the Euler totient function and

$$(3.5) \quad \Pi_q := \frac{\varphi(q)}{q} \prod_{p \nmid q} \left(1 - \frac{1}{p^2} \right).$$

Here $\rho(t)$ be the unique continuous solution of the difference-differential equation

$$(3.6) \quad \rho(t) = 1 \quad (0 \leq t \leq 1), \quad t\rho'(t) = -\rho(t-1) \quad (t > 1).$$

The implied constant in the O_U -symbol depends only on U .

We now introduce an auxiliary multiplicative function $h = h_{N_k, y}$ defined by

$$(3.7) \quad h_{N_k, y}(p) := \begin{cases} 0 & \text{if } p \mid N_k, \\ -2 - c\mathcal{L}^{-1} & \text{if } p > y \text{ and } p \nmid (N/2), \\ -c\mathcal{L}^{-1} & \text{if } \sqrt{y} < p \leq y \text{ and } p \nmid (N/2), \\ 1 - c\mathcal{L}^{-1} & \text{if } \mathcal{L}^2 \leq p \leq \sqrt{y} \text{ and } p \nmid (N/2), \end{cases}$$

and

$$(3.8) \quad h_{N_k, y}(p^\nu) := 0 \quad (\nu \geq 2),$$

where \mathcal{L} is defined as in (2.7) and the constant $c > 0$ will be chosen later.

The next lemma is the key for giving a lower bound of $S_f(x)$.

Lemma 3.2. *Let $k \geq 1$ be an integer and $N \geq 4$ an integer divisible by 4. Then for any $\varepsilon > 0$, we have, for $N_k \rightarrow \infty$,*

$$(3.9) \quad \sum_{n \leq y^u} h_{N_k, y}(n) \log \left(\frac{y^u}{n} \right) = \Pi_{N_k} y^u (\rho(2u) - 2 \log u) \left\{ 1 + O \left(\frac{1}{\log \mathcal{L}} \right) \right\}$$

uniformly for

$$(3.10) \quad 1 \leq u \leq \frac{3}{2} \quad \text{and} \quad (k^2 N)^{1/100} \leq y \leq (k^2 N)^2,$$

where Π_{N_k} and $\rho(u)$ are defined as in Lemma 3.1, and the implied constant in the O -symbol is absolute. In particular $\rho(2u) - 2 \log u > 0$ for all $u < \kappa$ where κ is the solution to $\rho(2\kappa) = 2 \log \kappa$. We have $\kappa > \frac{10}{9}$.

Proof. According to the definition of $h_{N_k, y}$, we have

$$(3.11) \quad \sum_{n \leq y^u} h_{N_k, y}(n) \log \left(\frac{y^u}{n} \right) = S_1 + O(\mathcal{L}^{-1} S_2) - (2 + c\mathcal{L}^{-1}) S_3,$$

for all u and y satisfying (3.10), where

$$\begin{aligned} S_1 &:= \sum_{\substack{n \leq y^u \\ P(n) \leq \sqrt{y}}} h_{N_k, y}(n) \log \left(\frac{y^u}{n} \right), \\ S_2 &:= \sum_{\substack{\sqrt{y} < p \leq y \\ p \nmid N_k}} \sum_{\substack{n \leq y^u/p \\ p \nmid n}} h_{N_k, y}(n) \log \left(\frac{y^u}{pn} \right), \\ S_3 &:= \sum_{\substack{y < p \leq y^u \\ p \nmid N_k}} \sum_{\substack{n \leq y^u/p}} h_{N_k, y}(n) \log \left(\frac{y^u}{pn} \right). \end{aligned}$$

For square-free $n \leq y^u \leq (k^2 N)^3$ with $P(n) \leq \sqrt{y}$ and $(n, N_k) = 1$, we have

$$\begin{aligned} h_{N_k, y}(n) &= \left(1 - \frac{c}{\mathcal{L}} \right)^{\omega(n)} = \exp \left\{ \omega(n) \log \left(1 - \frac{c}{\mathcal{L}} \right) \right\} \\ &= \exp \left\{ O \left(\frac{1}{\log \mathcal{L}} \right) \right\} = 1 + O \left(\frac{1}{\log \mathcal{L}} \right), \end{aligned}$$

where the implied constants are absolute. Thus

$$\begin{aligned} S_1 &= \left\{ 1 + O \left(\frac{1}{\log \mathcal{L}} \right) \right\} \sum_{n \leq y^u, (n, N_k) = 1, P(n) \leq \sqrt{y}} \mu(n)^2 \log \left(\frac{y^u}{n} \right) \\ &= \left\{ 1 + O \left(\frac{1}{\log \mathcal{L}} \right) \right\} \int_{1-}^{y^u} \log \left(\frac{y^u}{t} \right) d\Xi_{N_k}(t, \sqrt{y}) \\ &= \left\{ 1 + O \left(\frac{1}{\log \mathcal{L}} \right) \right\} \int_1^{y^u} \frac{\Xi_{N_k}(t, \sqrt{y})}{t} dt. \end{aligned}$$

Here the implied constants are absolute. By Lemma 3.1 with $(q, y, y^u) = (N_k, \sqrt{y}, t)$, it follows that

$$\int_1^{y^u} \frac{\Xi_{N_k}(t, \sqrt{y})}{t} dt = \left\{ 1 + O \left(\frac{\log^{e+2} \mathcal{L}}{\sqrt{\mathcal{L}}} \right) \right\} \Pi_{N_k} \int_1^{y^u} \rho \left(\frac{\log t}{\log \sqrt{y}} \right) dt,$$

where the implied constant is absolute. By making the change of variables $t = y^{u-v/2}$ and by partial integration, we deduce

$$\begin{aligned} \int_1^{y^u} \rho \left(\frac{\log t}{\log \sqrt{y}} \right) dt &= y^u (\log \sqrt{y}) \int_0^{2u} y^{-v/2} \rho(2u - v) dv \\ &= y^u \left\{ \rho(2u) - \left(\int_0^{1/\sqrt{\mathcal{L}}} + \int_{1/\sqrt{\mathcal{L}}}^{2u} \right) y^{-v/2} \rho'(2u - v) dv \right\} \\ &= y^u \rho(2u) \left\{ 1 + O \left(\frac{1}{\sqrt{\mathcal{L}}} \right) \right\}, \end{aligned}$$

where the implied constant is absolute. Combining these estimations, we find that

$$(3.12) \quad S_1 = \left\{ 1 + O \left(\frac{1}{\log \mathcal{L}} \right) \right\} y^u \Pi_{N_k} \rho(2u),$$

where the implied constant is absolute.

Similarly we can write

$$S_3 = S'_3 - S''_3,$$

where

$$\begin{aligned} S'_3 &= \left\{ 1 + O\left(\frac{1}{\log \mathcal{L}}\right) \right\} \sum_{y < p \leq y^u} \sum_{\substack{n \leq y^u/p \\ (n, N_k)=1}} \mu(n)^2 \log\left(\frac{y^u}{pn}\right) \\ &= \left\{ 1 + O\left(\frac{1}{\log \mathcal{L}}\right) \right\} \sum_{y < p \leq y^u} \int_{1-}^{y^u/p} \log\left(\frac{y^u}{pt}\right) d\Xi_{N_k}(t) \\ &= \left\{ 1 + O\left(\frac{1}{\log \mathcal{L}}\right) \right\} \sum_{y < p \leq y^u} \int_1^{y^u/p} \frac{\Xi_{N_k}(t)}{t} dt \\ &= \left\{ 1 + O\left(\frac{1}{\log \mathcal{L}}\right) \right\} y^u \Pi_{N_k} \sum_{y < p \leq y^u} \frac{1}{p} \\ &= \left\{ 1 + O\left(\frac{1}{\log \mathcal{L}}\right) \right\} y^u \Pi_{N_k} \log u, \end{aligned}$$

and

$$\begin{aligned} S''_3 &\ll \sum_{\substack{y < p \leq y^u \\ p|N_k}} \sum_{\substack{n \leq y^u/p \\ (n, N_k)=1}} \mu(n)^2 \log\left(\frac{y^u}{pn}\right) \ll y^u \Pi_{N_k} \sum_{\substack{y < p \leq y^u \\ p|N_k}} \frac{1}{p} \\ &\ll y^{u-1} \Pi_{N_k} \omega(N_k) \ll y^u \Pi_{N_k} (\log \mathcal{L})^{-1}, \end{aligned}$$

where we have used the following estimates : $\omega(N_k) \ll \log N_k \ll \mathcal{L}^2 \ll y(\log \mathcal{L})^{-1}$ since $y \geq (k^2 N)^{1/100}$. Here the implied constants are absolute. These imply that

$$(3.13) \quad S_3 = \left\{ 1 + O\left(\frac{1}{\log \mathcal{L}}\right) \right\} y^u \Pi_{N_k} \log u,$$

where the implied constant is absolute.

Finally we have

$$\begin{aligned} S_2 &\leq \sum_{\sqrt{y} < p \leq y} \sum_{n \leq y^u/p, (n, N_k)=1} \mu(n)^2 \log\left(\frac{y^u}{pn}\right) \\ &= \sum_{\sqrt{y} < p \leq y} \int_{1-}^{y^u/p} \log\left(\frac{y^u}{pt}\right) d\Xi_{N_k}(t) \\ (3.14) \quad &= \sum_{\sqrt{y} < p \leq y} \int_1^{y^u/p} \frac{\Xi_{N_k}(t)}{t} dt \\ &\ll y^u \Pi_{N_k} \sum_{\sqrt{y} < p \leq y} \frac{1}{p} \\ &\ll y^u \Pi_{N_k}, \end{aligned}$$

where the implied constants are absolute.

Inserting (3.12), (3.14) and (3.13) into (3.11), we get (3.9). \square

3.3. Lower bound for $S_{\mathfrak{f}}(x)$.

From (2.15), the inversion formula of Möbius allows us to deduce

$$(3.15) \quad \mathfrak{a}_{\mathfrak{f}}^*(n) = \sum_{d|n} \frac{\mu(d)\chi_{t,N}(d)}{\sqrt{d}} \lambda_f\left(\frac{n}{d}\right).$$

Taking $n = p^\nu$, it follows that

$$(3.16) \quad \mathfrak{a}_{\mathfrak{f}}^*(p^\nu) = \lambda_f(p^\nu) - \frac{\chi_{t,N}(p)}{\sqrt{p}} \lambda_f(p^{\nu-1}).$$

Thus

$$(3.17) \quad \mathfrak{a}_{\mathfrak{f}}^*(p^\nu) \geq 0 \Leftrightarrow \lambda_f(p^\nu) \geq \frac{\chi_{t,N}(p)}{\sqrt{p}} \lambda_f(p^{\nu-1}) \Rightarrow \lambda_f(p^\nu) \geq -\frac{\nu}{\sqrt{p}},$$

where we have used the Deligne bound.

Since f is a newform of weight $2k$ for the group $\Gamma_0(N/2)$ with character χ^2 , for each prime $p \nmid (N/2)$ there is a unique real $\theta_f(p) \in [0, \pi]$ such that

$$(3.18) \quad \lambda_f(p^\nu) = \frac{\sin((\nu+1)\theta_f(p))}{\sin \theta_f(p)} \quad (\nu \geq 1).$$

Let $y_{\mathfrak{f}} > 0$ be the largest integer such that

$$(3.19) \quad \mathfrak{a}_{\mathfrak{f}}^*(n) \geq 0 \quad \text{for } n \leq y_{\mathfrak{f}} \text{ and } (n, N/2) = 1.$$

We now proceed to establish a lower bound for $S_{\mathfrak{f}}(x)$ by using the assumption of positivity (3.19). For primes $\mathcal{L}^2 \leq p \leq y_{\mathfrak{f}}$ with $p \nmid (N/2)$, we thus have $\mathfrak{a}_{\mathfrak{f}}(tp^2) \geq 0$. From (3.17) and (3.18) with $\nu = 1$, it follows that

$$\lambda_f(p) = \frac{\sin(2\theta_f(p))}{\sin \theta_f(p)} \geq -\frac{1}{\sqrt{p}} \geq -\frac{2}{\mathcal{L}}.$$

Hence, there is an absolute positive constant $c_1 > 0$ such that $\theta_f(p) \leq \frac{\pi}{2} + \frac{c_1}{\mathcal{L}}$. Furthermore, if $\mathcal{L}^2 \leq p \leq \sqrt{y_{\mathfrak{f}}}$ and $p \nmid (N/2)$, we have

$$\lambda_f(p^2) = \frac{\sin(3\theta_f(p))}{\sin \theta_f(p)} \geq -\frac{2}{\sqrt{p}} \geq -\frac{2}{\mathcal{L}}.$$

This implies that $\theta_f(p) \leq \frac{\pi}{3} + \frac{c'}{\mathcal{L}}$ and

$$\lambda_f(p) \geq 2 \cos\left(\frac{\pi}{3} + \frac{c'}{\mathcal{L}}\right) \geq 1 - \frac{c}{2\mathcal{L}},$$

where $c > 0$ is an absolute positive constant.

In view of (3.16) with $\nu = 1$ and the definition of $h_{N_k, y_{\mathfrak{f}}}(p)$, it is clear that

$$(3.20) \quad \mathfrak{a}_{\mathfrak{f}}^*(p) \geq h_{N_k, y_{\mathfrak{f}}}(p)$$

for all prime numbers $p \nmid N_k$.

Proposition 2. *Let $k \geq 1$ be an integer, $N \geq 4$ an integer divisible by 4 and χ be a real Dirichlet character modulo N . Suppose that $\mathfrak{f} \in \mathcal{S}_{k+1/2}^*(N, \chi)$ is a Hecke eigenform and $t \geq 1$ is a square-free integer such that $\mathfrak{a}_{\mathfrak{f}}(t) > 0$. Assume that its Shimura lift is not of CM type and that $(k^2 N)^{1/100} \leq y_{\mathfrak{f}} \leq (k^2 N)^2$. Then we have*

$$(3.21) \quad S_{\mathfrak{f}}(y_{\mathfrak{f}}^u) \gg y_{\mathfrak{f}}^u (\log_2(kN))^{-1}$$

for all $u < \kappa$, where $y_{\mathfrak{f}}$ and κ are defined as in (3.19) and Lemma 3.2, respectively, and \log_r means the r -fold iterated logarithm. Here the implied constant is absolute.

Proof. With the help of (2.5), it is easy to verify that $n \mapsto \mathfrak{a}_{\mathfrak{f}}^*(n)$ is multiplicative. Let $g_{N_k, y_{\mathfrak{f}}}$ be the multiplicative function defined by the Dirichlet convolution identity $\mathfrak{a}_{\mathfrak{f}}^* \mu^2 = g_{N_k, y_{\mathfrak{f}}} * h_{N_k, y_{\mathfrak{f}}}$. Then $g_{N_k, y_{\mathfrak{f}}}(n) \geq 0$ for all square-free integers $n \geq 1$ with $(n, N_k) = 1$, since $g_{N_k, y_{\mathfrak{f}}}(p) = \mathfrak{a}_{\mathfrak{f}}^*(p) - h_{N_k, y_{\mathfrak{f}}}(p) \geq 0$ for $p \nmid N_k$ thanks to (3.20).

On the other hand, according to Lemma 3.2, we have

$$\sum_{n \leq y_{\mathfrak{f}}^u} h_{N_k, y_{\mathfrak{f}}}(n) \log \left(\frac{y_{\mathfrak{f}}^u}{n} \right) \geq 0$$

for $u < \kappa$ and sufficiently large $y_{\mathfrak{f}}$. But, as $g_{N_k, y_{\mathfrak{f}}}(1) = 1$, we infer that

$$\begin{aligned} S_{\mathfrak{f}}(y_{\mathfrak{f}}^u) &= \sum_{n \leq y_{\mathfrak{f}}^u} g_{N_k, y_{\mathfrak{f}}} * h_{N_k, y_{\mathfrak{f}}}(n) \log \left(\frac{y_{\mathfrak{f}}^u}{n} \right) \\ &= \sum_{d \leq y_{\mathfrak{f}}^u} g_{N_k, y_{\mathfrak{f}}}(d) \sum_{m \leq y_{\mathfrak{f}}^u/d} h_{N_k, y_{\mathfrak{f}}}(m) \log \left(\frac{y_{\mathfrak{f}}^u}{dm} \right) \\ &\geq \sum_{m \leq y_{\mathfrak{f}}^u} h_{N_k, y_{\mathfrak{f}}}(m) \log \left(\frac{y_{\mathfrak{f}}^u}{m} \right). \end{aligned}$$

Then we get the required lower bound (3.21) by Lemma 3.2, since we have, by (3.5) and the prime number theorem,

$$\Pi_{N_k} \gg \frac{\varphi(N_k)}{N_k} \gg (\log_2 N_k)^{-1} \gg \{\log_2(kN)\}^{-1},$$

where the implied constant is absolute. This completes the proof. \square

3.4. End of the proof of Theorem 1.

Without loss of generality, we can assume that $y_{\mathfrak{f}} \geq (k^2 N)^{1/100}$.

Firstly we prove

$$(3.22) \quad y_{\mathfrak{f}} \ll (k^2 N)^2,$$

where the implied constant is absolute.

By the definition of $y_{\mathfrak{f}}$, the multiplicativity of $\mathfrak{a}_{\mathfrak{f}}^*(n)$ and (3.20), we have

$$\begin{aligned}
S_{\mathfrak{f}}(y_{\mathfrak{f}}) &\geq \sum_{\substack{\mathcal{L}^2 \leq p \neq p' \leq y_{\mathfrak{f}}^{1/3} \\ (pp', N/2)=1}} \mathfrak{a}_{\mathfrak{f}}^*(pp') \log \left(\frac{y_{\mathfrak{f}}}{pp'} \right) \\
&\geq 3(\log y_{\mathfrak{f}}) \sum_{\substack{\mathcal{L}^2 \leq p \neq p' \leq y_{\mathfrak{f}}^{1/3} \\ (pp', N/2)=1}} \left(1 - \frac{c}{2\mathcal{L}} \right)^2 \\
&\gg (\log y_{\mathfrak{f}}) \left\{ \left(\sum_{\substack{\mathcal{L}^2 \leq p \leq y_{\mathfrak{f}}^{1/3} \\ (p, N/2)=1}} 1 \right)^2 - \sum_{p \leq y_{\mathfrak{f}}^{1/3}} 1 \right\} \\
&\gg y_{\mathfrak{f}}^{2/3},
\end{aligned}$$

where the implied constant is absolute. Combining this with Proposition 1 yields

$$y_{\mathfrak{f}}^{2/3} \ll S_{\mathfrak{f}}(y_{\mathfrak{f}}) \ll_{\varepsilon} (k^2 N)^{1/4+\varepsilon} y_{\mathfrak{f}}^{1/2}$$

where the first implied constant is absolute and the second depends only on ε . Clearly these imply the required inequality (3.22).

In view of (3.22), we can apply Propositions 1 and 2 to write

$$(3.23) \quad y_{\mathfrak{f}}^u (\log \mathcal{L})^{-1} \ll S_{\mathfrak{f}}(y_{\mathfrak{f}}^u) \ll_{\varepsilon} (k^2 N)^{1/4+\varepsilon} y_{\mathfrak{f}}^{u/2}$$

for $u < \kappa$, where the first implied constant is absolute and the second depends only on ε . From (3.23) we deduce that $y_{\mathfrak{f}} \ll_{\varepsilon} (k^2 N)^{1/(2u)+\varepsilon}$ for $1 \leq u < \kappa$. According to Lemma 3.2, we know $\kappa > \frac{10}{9}$. Thus $y_{\mathfrak{f}} \ll (k^2 N)^{9/20}$, where the implied constant is absolute. This is equivalent to the result of Theorem 1.

4. PROOF OF THEOREM 2

Consider the function

$$\varepsilon_{\mathfrak{f}}(n) = \text{sign } \mathfrak{a}_{\mathfrak{f}}(tn^2) := \begin{cases} 1 & \text{si } \mathfrak{a}_{\mathfrak{f}}(tn^2) > 0, \\ -1 & \text{si } \mathfrak{a}_{\mathfrak{f}}(tn^2) < 0, \\ 0 & \text{si } \mathfrak{a}_{\mathfrak{f}}(tn^2) = 0. \end{cases}$$

By using (2.5), it is easy to check that this function is multiplicative. According to [6, Theorem], for any real multiplicative function such that $|g(n)| \leq 1$, the inequality

$$\sum_{n \leq x} g(n) \ll x \exp \left(-K \sum_{p \leq x} \frac{1 - g(p)}{p} \right)$$

holds for all $x \geq 2$, where $K = 0.32867 \dots = -\cos \phi_0$ and ϕ_0 is the unique root in $(0, \pi)$ of the equation $\sin \phi - \phi \cos \phi = \frac{1}{2}\pi$. Applying this result to $\varepsilon_{\mathfrak{f}}(n)$, it follows

that

$$\begin{aligned} \sum_{n \leq x} \varepsilon_{\mathfrak{f}}(n) &\ll x \exp \left(-K \sum_{p \leq x} \frac{1 - \varepsilon_{\mathfrak{f}}(p)}{p} \right) \\ &\ll x \exp \left(-K \sum_{p \leq x, \varepsilon_{\mathfrak{f}}(p) = -1} \frac{2}{p} \right). \end{aligned}$$

With the help of Lemma 2.2, a simple integration by parts gives us

$$\sum_{p \leq x, \varepsilon_{\mathfrak{f}}(p) = -1} \frac{1}{p} = \int_{2-}^x \frac{1}{t} d \left(\sum_{p \leq t, \varepsilon_{\mathfrak{f}}(p) = -1} 1 \right) \geq \left(\frac{1}{2} - \varepsilon \right) \log_2 x$$

for any $\varepsilon > 0$ and all $x \geq x_0(\mathfrak{f}, \varepsilon)$.

Combining these two estimates, we find that

$$(4.1) \quad \sum_{n \leq x} \varepsilon_{\mathfrak{f}}(n) \ll_{\mathfrak{f}, \varepsilon} \frac{x}{(\log x)^{K-\varepsilon}}$$

for any $\varepsilon > 0$ and all $x \geq 2$.

On the other hand, a particular case of [15, Theorem 2] can be stated as follows: Let h be a non-negative multiplicative function satisfying the following conditions

$$(4.2) \quad \sum_{p \leq z} h(p) \log p = \kappa z + O \left(\frac{z}{(\log z)^\delta} \right) \quad (z \geq 2),$$

$$(4.3) \quad \sum_{p, \nu \geq 2} \frac{h(p^\nu)}{p^\nu} \log p^\nu \leq A,$$

where $A > 0$, $\kappa > 0$ and $\delta > 0$ are constants. Then we have

$$(4.4) \quad \sum_{n \leq x} h(n) = C_h x (\log x)^{\kappa-1} \left\{ 1 + O_{h, \delta} \left(\frac{\log_2 x}{\log x} + \frac{1}{(\log x)^\delta} \right) \right\},$$

where

$$C_h := \prod_p (1 - p^{-1})^\kappa \sum_{\nu \geq 0} g(p^\nu) p^{-\nu}.$$

In view of Lemma 2.3 and the prime number theorem, we immediately see that

$$\sum_{p \leq z} |\varepsilon_{\mathfrak{f}}(p)| \log p = z + O_{\mathfrak{f}, \varepsilon} (z (\log z)^{-1/4+\varepsilon}) \quad (z \geq 2).$$

This shows that the non-negative multiplicative function $n \mapsto |\varepsilon_{\mathfrak{f}}(n)|$ satisfies the condition (4.2). Since $|\varepsilon_{\mathfrak{f}}(n)| \leq 1$ for all $n \geq 1$, the condition (4.3) is verified trivially. Thus (4.4) implies that

$$(4.5) \quad \sum_{n \leq x} |\varepsilon_{\mathfrak{f}}(n)| = \rho_{\mathfrak{f}} x \left\{ 1 + O_{\mathfrak{f}, \varepsilon} ((\log x)^{-1/4+\varepsilon}) \right\}.$$

This is equivalent to the assertion (i).

Notice that

$$\frac{|\varepsilon_{\mathfrak{f}}(n)| + \varepsilon_{\mathfrak{f}}(n)}{2} = \begin{cases} 1 & \text{if } \varepsilon_{\mathfrak{f}}(n) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The estimates (4.1) and (4.5) imply that

$$(4.6) \quad \sum_{\substack{n \leq x \\ \varepsilon_f(n)=1}} 1 = \frac{1}{2} \rho_f x \{1 + O_{f,\varepsilon}((\log x)^{-1/4+\varepsilon})\}.$$

This is equivalent to the assertion (ii) with sign $+$. The other case can be treated in the same way.

5. PROOF OF THEOREM 3

In order to generalize the square-free numbers, Erdős [5] introduced the notion of \mathcal{B} -free numbers. More precisely, let

$$\mathcal{B} = \{b_i : 1 < b_1 < b_2 < \dots\}$$

be a sequence of integers verifying the following conditions

$$(5.1) \quad \sum_{i \geq 1} b_i^{-1} < \infty$$

and

$$(5.2) \quad (b_i, b_j) = 1 \quad (i \neq j).$$

A positive integer $n \geq 1$ is called \mathcal{B} -free if it is not divisible by any element in \mathcal{B} . Many authors studied the distribution of \mathcal{B} -free integers. A detailed historical description can be found in [11, 22]. In particular, by using sieve and estimates for multiple exponential sums, the authors of these two papers proved the following results (see [11, Corollary 10] and [22, Proposition 2], respectively):

- For all $\varepsilon > 0$, $x \geq x_0(\mathcal{B}, \varepsilon)$ and $y \geq x^{7/17+\varepsilon}$, we have

$$(5.3) \quad |\{x < n \leq x + y : n \text{ is } \mathcal{B}\text{-free}\}| \gg_{\mathcal{B}, \varepsilon} y.$$

- For all $\varepsilon > 0$, $x \geq x_0(\mathcal{B}, \varepsilon)$, $y \geq x^{17/38+100\varepsilon}$, $1 \leq a \leq q \leq x^\varepsilon$ with $((a, q), b) = 1$ for all $b \in \mathcal{B}$, we have

$$(5.4) \quad |\{x < n \leq x + y : n \equiv a \pmod{q} \text{ and } n \text{ is } \mathcal{B}\text{-free}\}| \gg_{\mathcal{B}, \varepsilon} y/q.$$

Here the implied constants depend only on \mathcal{B} and ε .

Now let \mathcal{P} be the set of all primes and define

$$\mathfrak{P}_f := \{p \in \mathcal{P} : \mathfrak{a}_f(tp^2) = 0\}, \quad \mathcal{B}_f := \mathfrak{P}_f \cup \{p^2 : p \in \mathcal{P} \setminus \mathfrak{P}_f\}.$$

Clearly if n is \mathcal{B}_f -free, then n is square-free and by all its prime factors are not in \mathfrak{P}_f . By using (2.5), it is easy to see that $\mathfrak{a}_f(tn^2) \neq 0$ for all \mathcal{B}_f -free numbers n . Thus (5.3) and (5.4) imply the first and second assertions of Theorem 3, respectively, if we can show that the sequence \mathcal{B}_f verifies the conditions (5.1) and (5.2).

Firstly, the definition of \mathcal{B}_f guarantees that the condition (5.2) is satisfied by \mathcal{B}_f . On the other hand, in view of (2.1), we have

$$a_{ft}(p) = \mathfrak{a}_f(tp^2) + \chi_{t,N}(p) \mathfrak{a}_f(t) p^{k-1}.$$

Thus (2.22) of Lemma 2.3 implies that

$$\sum_{\substack{p \leq x \\ \mathfrak{a}_{\mathfrak{f}}(tp^2)=0}} 1 = \sum_{\substack{p \leq x \\ \mathfrak{a}_{f_t}(p)=\chi_{t,N}(p)\mathfrak{a}_{\mathfrak{f}}(t)p^{k-1}}} 1 \ll_{\mathfrak{f}} \frac{x}{(\log x)^{5/4-\varepsilon}},$$

where the implied constant depends on \mathfrak{f} . Hence a simple integration by parts gives us

$$\begin{aligned} \sum_{p \leq x, p \in \mathfrak{P}_{\mathfrak{f}}} \frac{1}{p} &= \int_{2-}^x \frac{1}{t} d\left(\sum_{p \leq t, \mathfrak{a}_{\mathfrak{f}}(tp^2)=0} 1 \right) \\ &= \frac{1}{x} \sum_{p \leq x, \mathfrak{a}_{\mathfrak{f}}(tp^2)=0} 1 + \int_2^x \frac{1}{t^2} \left(\sum_{p \leq t, \mathfrak{a}_{\mathfrak{f}}(tp^2)=0} 1 \right) dt \\ &\ll_{\mathfrak{f}} \frac{1}{(\log x)^{5/4-\varepsilon}} + \int_2^x \frac{dt}{t(\log t)^{5/4-\varepsilon}} \\ &\ll_{\mathfrak{f}} 1, \end{aligned}$$

where the implied constant depends on \mathfrak{f} . This implies that the sequence $\mathcal{B}_{\mathfrak{f}}$ verifies the condition (5.1).

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BIN CHEN, DEPARTMENT OF MATHEMATICS, TONGJI UNIVERSITY, SHANGHAI 200092, CHINA.
 DEPARTMENT OF MATHEMATICS, WEINAN NORMAL UNIVERSITY, WEINAN 714000, CHINA.
E-mail address: 13tjccbb@tongji.edu.cn

JIE WU, SCHOOL OF MATHEMATICS, SHANDONG UNIVERSITY, JINAN, SHANDONG 250100, CHINA. CNRS, INSTITUT ÉLIE CARTAN DE LORRAINE, UMR 7502, 54506 VANDŒUVRE-LÈS-NANCY, FRANCE

Current address: Université de Lorraine, Institut Élie Cartan de Lorraine, UMR 7502, 54506 Vandœuvre-lès-Nancy, France

E-mail address: jie.wu@univ-lorraine.fr